KMS, ETC.

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4. Applications to QFT
5. Perhaps, a little funct. analysis — or perhaps not.
6. Conclusions.

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1. Introduction, résumé

Goal of exercise: General theory of q.m. matter in thermal equilibrium — approach based on (imag.-time) T-O-G-F's. ($\leftrightarrow$ functional integrals)

$\longleftrightarrow$ Phase trans. & crit. phenomena (s.c., magn.), some transport phen., cosmology, (CMWB ...).

KMS (Boundary) Cond., KMS states

$\updownarrow$

PCT theorem; spin & statistics; QFT in non-trivial grav. backgrounds, such as

- (Schwartzschild) black hole
- de Sitter space
- AdS; CFT

Connections to topics in "NC geometry". ("Remake")

1974 – 2002

2. KMS states, according to H-H-W & Araki

i) Thermal equilibrium, inv. temperature β, chem. pot. μ

Equilibrium state is

$$\text{"}\,\Xi_{\beta,\mu}^{-1}\,\mathrm{Tr}\left[\exp -\beta(H-\mu Q)\,(\cdot)\right]\text{"}$$

(L. - v. N)

$$\Downarrow$$

$$\langle a\,\alpha_t(b)\rangle_{\beta,\mu} = \langle \alpha_{t-i\beta}(b)\,a\rangle_{\beta,\mu},$$

$$\alpha_t(b) := e^{itH}\, b\, e^{-itH}.$$

ii) In thermodynamic limit, describe system as a

<u>C^*-dynamical system.</u>

C^*-algebra $\mathcal{A}$ of "<u>obs.</u>"

<u>States</u> $\Leftrightarrow$ pos., linear functs., ω, on $\mathcal{A}$, $\omega(\mathbb{1}) = 1$.

<u>Symmetries</u> $\leftrightarrow$ *automor. groups of $\mathcal{A}$

E.g. time translations

$$\{\alpha_t \mid t \in \mathbb{R}\}$$

(time-transl. inv. states...)

iii) State ω_β is β-<u>KMS state</u> for $\{\alpha_t\}$ iff

$$\omega_\beta(a\,\alpha_t(b)) = \omega_\beta(\alpha_{-i\beta+t}(b)a)$$

$$\left[= \omega_\beta(b\,\alpha_{i\beta-t}(a))\right]$$

$\omega_\beta(a\alpha_t(b))$ is b.v. of function $F^\beta_{ab}(z)$

analytic in z on

$$0 < \mathrm{Im}\, z < \beta$$

$\Rightarrow \omega_\beta$ time transl. inv.

iv) <u>GNS construction</u>

$(\mathcal{A}, \alpha_t, \omega_\beta)$ gives rise to

$\mathcal{H}_\beta$, rep. λ of $\mathcal{A}$, cyclic & separating vector $\Omega_\beta \in \mathcal{H}_\beta$, unitary one-param. group $\{e^{it\mathcal{L}} \mid t \in \mathbb{R}\}$

such that

$$\lambda(\alpha_t(a)) = e^{it\mathcal{L}} \lambda(a) e^{-it\mathcal{L}},$$

$$e^{it\mathcal{L}} \Omega_\beta = \Omega_\beta,$$

$$\langle \Omega_\beta, \lambda(a)\Omega_\beta \rangle = \omega_\beta(a)$$

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v) Modular conjugation.

Ω_β cyclic (GNS) & separating (KMS)

$\Rightarrow \quad S\lambda(a)\Omega_\beta = \lambda(a)^* \Omega_\beta, \; a \in \mathcal{A},$

def. (closable) op. S.

($\leftrightarrow$ Tomita-Takesaki)

$$J\lambda(a)\Omega_\beta := S\lambda(\alpha_{-i\beta/2}(a))\Omega_\beta$$

$$= \lambda(\alpha_{i\beta/2}(a^*))\Omega_\beta$$

Lemma. J is **anti-unit.**
(use KMS cond.!)

$$S = J e^{-\beta \mathcal{L}/2} = e^{\beta \mathcal{L}/2} J$$

Def. $\rho(a) := J \lambda(a) J$, $a \in \mathcal{A}$
is anti-rep. of $\mathcal{A}$ **comm.**
with λ: $\rho[\mathcal{A}]'' = \lambda[\mathcal{A}]'$ $*$

$$J e^{it\mathcal{L}} = e^{it\mathcal{L}} J, \quad \forall t$$

$$(\Leftrightarrow \quad J \mathcal{L} = -\mathcal{L} J)$$

- $J \leftrightarrow PCT$, $* \leftrightarrow$ locality
- Tomita-Takesaki th.
- Return to equilibrium.

vi) Thermal Green fus.

$$F^\beta(t_1,\dots,t_n) := \omega_\beta\left(\prod_{j=1}^{n} \alpha_{t_j}(a_j)\right)$$

$\rightarrow$ Correlations, phase trans., (neutron scatt.)

F^β only depends on $t_{i+1}-t_i, \dots$

KMS . . . . (p. 13):

$$\Downarrow$$

F^β boundary value of

$$F^\beta(t_1,\dots,t_j,t_{j+1}+z,\dots,t_n+z),$$

$$0 < \operatorname{Im} z < \beta$$

$\hookrightarrow$ Apply Malgrange-Zerner theorem:

$F^\beta(z_1, \dots, z_n)$ analytic in $z_1, \dots, z_n$ on tube

$$\mathcal{T}_n := \{z_1, \dots, z_n \mid \operatorname{Im} z_i < \operatorname{Im} z_{i+1}, \; \operatorname{Im}(z_n - z_1) < \beta\},$$

$$|F^\beta(z_1, \dots, z_n)| \leq \prod \|a_j\|.$$

"Temperature-ordered Green functions" (T-O-G-F)

$$\varphi^\beta(a_1, \tau_1, \dots, a_n, \tau_n) := F^\beta(i\tau_1, \dots, i\tau_n),$$

$$\tau_1 < \tau_2 < \dots < \tau_n, \; \tau_n - \tau_1 < \beta.$$

Properties of T-O-G-F

(1) $\varphi^\beta(a_1, \tau_1, \dots, a_n, \tau_n)$ **lin.** in a_j, $\forall j$, **cont.** in $\tau_1 < \tau_2 < \dots < \tau_n < \tau_1 + \beta$

(2) $$\varphi^\beta(a_1, \tau_1, \dots, a_n, \tau_n) = \varphi^\beta(a_1, \tau_1 + \tau, \dots, a_n, \tau_n + \tau)$$

(3) **KMS condition:**

$$\varphi^\beta(a_1, \tau_1, \dots, a_n, \tau_n) = \varphi^\beta(a_{j+1}, \tau_{j+1}, \dots, a_n, \tau_n, a_1, \tau_1 + \beta, \dots, a_j, \tau_j + \beta)$$

(4) Reflection Positivity:

$$M_{ij} := \varphi^\beta(a_1^i, \tau_1^i, \dots, a_{n_i}^i, \tau_{n_i}^i, (a_{n_j}^j)^*, \beta - \tau_{n_j}^j, \dots (a_1^j)^*, \beta - \tau_1^j),$$

$$0 < \tau_1^i < \dots < \tau_{n_i}^i < \frac{\beta}{2}, \quad \forall i$$

def. positive semi-def. matrix $M = (M_{ij})$

(5) φ^β determined by its values on countbl. set of $[a_1, \tau_1, \dots, a_n, \tau_n]$

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(6) a) $\varphi^\beta(a_1,\tau_1,\cdots,a_n,\tau_n)$

cont. on $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq \tau_1 + \beta$

b) $\varphi^\beta(a_1,\tau_1,\cdots,a_j,\tau_j,a_{j+1},\tau_j,\cdots)$

$= \varphi^\beta(a_1,\tau_1,\cdots,a_j\cdot a_{j+1},\tau_j,\cdots),$

$\forall j$

c) $\varphi^\beta(a_1,\tau_1,\cdots,\mathbb{1},\tau_j,\cdots a_n,\tau_n)$

$= \varphi^\beta(a_1,\tau_1,\cdots,\check{\tau}_j,\cdots a_n,\tau_n)$

d) $|\varphi^\beta(a_1,\tau_1,\cdots,a_n,\tau_n)|$

$\leq \prod_{j=1}^{n} \|a_j\|$ #

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(1)-(6) is a Theorem!

A priori, only T-O-G-F calculable from funct. integrals; KMS for FI

$\leftrightarrow$ <u>Matsubara</u>:

$$\langle \Psi^*(\tau_1)\Psi(\tau_2)\rangle_\beta$$

$$\overset{KMS}{=} \langle \Psi(\tau_2)\Psi^*(\tau_1+\beta)\rangle_\beta$$

$$\overset{FI}{=} \begin{cases} \langle \Psi^*(\tau_1+\beta)\Psi(\tau_2)\rangle_\beta & \text{Bosons} \\ -\langle \Psi^*(\tau_1+\beta)\Psi(\tau_2)\rangle_\beta & \text{Fermions} \end{cases}$$

$$\Downarrow$$

Bosons $\leftrightarrow$ periodic b.c.

Fermions $\leftrightarrow$ anti-periodic b.c.

3. "Reconstruction Theorem"

Starting point:

Topol. vector space $\mathcal{A}$ (e.g. C^*-algebra); multi-linear functionals

$$\{\varphi_\beta(a_1,\tau_1,\dots,a_n,\tau_n)\}_{n=0}^{\infty},$$

$\varphi_\beta(a_1,\tau_1,\dots,a_n,\tau_n)$ def. for $a_1,\dots,a_n$ in $\mathcal{A}$, $\tau_1<\tau_2<\dots<\tau_n<\tau_1+\beta$, linear in each a_j, cont. in $\tau_1,\dots,\tau_n$.

Assume that φ_β's have properties (1)–(5).

Reconstruction theorem

(Gen. "GNS construction")

$\{\varphi_\beta\} \Longrightarrow \mathcal{H}_\beta, \Omega_\beta, \{e^{it\mathcal{L}}\}$,

anti-unitary op. J;

if $\mathcal{A}$ a *algebra $\Longrightarrow$

2 commuting reps.

λ and ρ (anti-rep.)

of $\mathcal{A}$ on $\mathcal{H}_\beta$;

$$\omega_\beta(\cdot) := \langle \Omega_\beta, (\cdot)\, \Omega_\beta \rangle$$

is β-KMS for

$\lambda[\mathcal{A}]$ w. time-evolution

$$\lambda(a) \longmapsto \underline{e^{it\mathcal{L}}} \lambda(a) e^{-it\mathcal{L}}$$

If $\{\varphi_\beta\}$ also has

<u>property</u> (6) then

determine <u>real-time</u>

<u>Green functions</u>

$$\left\{ \left\langle \Omega_\beta, \prod_{j=1}^{n} e^{it_j\mathcal{L}} \lambda(a_j) e^{-it_j\mathcal{L}} \Omega_\beta \right\rangle \right\}$$

satisf. β-<u>KMS Cond.</u>,

by <u>analytic cont.</u>

History: <u>Ruelle</u>, Høegh-K,

<u>O-S</u>, <u>J.F.</u>, K-L, ◆ '70's

A. Construction of $\mathcal{H}_\beta$, Ω_β:

$$\mathcal{V} = \left\langle [a_1, \tau_1, \dots, a_n, \tau_n] \,\middle|\, a_j \in \mathcal{A},\ 0 \leq \tau_1 < \dots < \tau_n < \frac{\beta}{2},\ n \in \mathbb{Z}_+ \right\rangle$$

linear space; equipped with pos. semi-def. inner product:

$$\langle [a_1, \tau_1, \dots, a_n, \tau_n], [b_1, \sigma_1, \dots, b_m, \sigma_m] \rangle$$

$$:= \varphi^\beta(a_1, \tau_1, \dots, a_n, \tau_n, b_m^*, \beta - \sigma_m, \dots, b_1^*, \beta - \sigma_1)$$

By prop. (4) ("RP"), pos. semi-def.; kernel $\mathcal{N}$;

Thermal Green functions

$$\langle \Omega_\beta, \prod_{j=1}^{n} e^{it_j L} \ell(a_j) e^{-it_j L} \Omega_\beta \rangle$$

determined by $\varphi^\beta(a_1, \tau_1, \cdots, a_n, \tau_n)$ by analytic continuation in time arguments & KMS!

History: Ruelle, Høegh-Krohn, Osterwalder-Schrader ($\beta = \infty$), J.F., Klein-Landau, F-O-S, L. Birke & J.F.

A. Construction of $\mathcal{H}_\beta, \Omega_\beta$:

$$\mathcal{V} := \langle [a_1, \tau_1, \cdots, a_n, \tau_n] \,|\, a_j \in \mathcal{A},$$
$$0 < \tau_1 < \cdots < \tau_n < \frac{\beta}{2},\ n \in \mathbb{Z}_+ \rangle$$

linear space equipped w.

$$\mathcal{D} := \mathcal{V}/_{\mathcal{N}}$$

$$v \in \mathcal{V} \longmapsto \Phi(v) := v \bmod \mathcal{N}$$

$$\mathcal{H}_\beta := \overline{\mathcal{D}}^{\langle\cdot,\cdot\rangle}$$

$$\Omega_\beta := \Phi(\emptyset) \qquad (\varphi^\beta(\emptyset) = 1)$$

B. Time translations

For $v = [a_1, \tau_1, \cdots, a_n, \tau_n] \in \mathcal{V}$,

$$v_\tau := [a_1, \tau_1 + \tau, \cdots a_n, \tau_n + \tau] \in \mathcal{V}$$

for $\varepsilon_-(v) < \tau < \varepsilon_+(v)$.

$$\langle v_\tau, w \rangle = \langle v, w_\tau \rangle$$

by prop. (1) (hermiticity)

$\mathcal{N}$ inv. under $v \mapsto v_\tau$,
by Schwarz $\Rightarrow$ define

$$\Gamma_\tau \Phi(v) := \Phi(v_\tau)$$

"local one-param. group"
on $\mathcal{D}$ with props. ....

Theorem

Γ_τ unique s.a. extension

$$e^{\tau \mathcal{L}}, \quad \mathcal{L} = \mathcal{L}^*,$$

with explicit "core".

C. Modular conjugation

$$J\Phi[a_1, \tau_1, \dots, a_n, \tau_n]$$
$$:= \Phi[a_n^*, \tfrac{\beta}{2} - \tau_n, \dots, a_1^*, \tfrac{\beta}{2} - \tau_1]$$

Using KMS & transl. inv. (p. 24): J anti-unitary,

$$J e^{\tau \mathcal{L}} = e^{-\tau \mathcal{L}} J,$$

$$\Rightarrow J e^{it\mathcal{L}} = e^{it\mathcal{L}} J$$

D. Left- & right reps. of $\mathcal{A}$.

Also assume prop. (6), with $\mathcal{A}$ a C^* algebra.

$$\lambda(a)\,\Phi[a_1,\tau_1,\dots,a_n,\tau_n]$$

$$:= \Phi[a,0,a_1,\tau_1,\dots,a_n,\tau_n]$$

$$\rho(a)\,\Phi[a_1,\tau_1,\dots,a_n,\tau_n]$$

$$:= \Phi[a_1,\tau_1,\dots,a_n,\tau_n,a^*,\tfrac{\beta}{2}]$$

$$\rho(a) = J\lambda(a)J$$

$\mathcal{H}_\beta$ is $\mathcal{A}$-bimodule

Lemma.

1) $\|\lambda(a)\|, \|\rho(a)\| \leq \|a\|$

2) $[e^{it\mathcal{L}}\lambda(a)e^{-it\mathcal{L}}, e^{is\mathcal{L}}\rho(b)e^{-is\mathcal{L}}] = 0$ (p. 27)

E. Analytic Cont. of T-O-G-F

$\mathcal{A}, \{\varphi^\beta\}$ with props. (1)–(6).

$\varphi^\beta(a_1, \tau_1, \dots, a_n, \tau_n)$ is restr. of $F^\beta(a_1, z_1, \dots, a_n, z_n)$ **holo.** on

$$\mathcal{T}_n^{(\beta)} := \{(z_1, \dots, z_n) \mid Im\, z_1 < \dots < Im\, z_n < Im\, z_1 + \beta\}$$

to $\{z_j = i\tau_j, \tau_j \in \mathbb{R}\}$;

& KMS Condition! (→ p. 37)

$$\Rightarrow F^\beta(a_1, t_1, \dots, a_n, t_n)$$

$$= \left\langle \Omega_\beta, \prod_{j=1}^{n} e^{it_j \mathcal{L}} \lambda(a_j) e^{-it_j \mathcal{L}} \Omega_\beta \right\rangle$$

real-time Green fu.

Proof by induction, using Lemma + convexity; generalizations to *alg. of unbounded ops.?

K & L ('80); L. B. & J. F.

4. Applications to QFT

Local, relat. QFT; vacuum Ω; $\mathcal{M}$: generator of a boost; $\mathcal{A}$: topol. vector space ("time-0 test fns."); field operators $\Psi(f)$;

<u>covariance</u>:

$$e^{\alpha \mathcal{M}} \Psi(f,0) e^{-\alpha \mathcal{M}} = \Psi(S(\alpha)f, \alpha)$$

α: imag. boost angle; $S(\alpha)$ rep. of $SO(d)$ on $\mathcal{A}$.

$\varphi(f_1, \alpha_1, \cdots, f_n, \alpha_n)$:

Schwinger fns.

$\sim$ T-O-G-F's

Here, reconstr. thm. due to O-S, Glaser.

$$0 \le \langle e^{\alpha M} \Phi(f,0)\Omega, e^{\alpha M} \Phi(f,0)\Omega \rangle$$

$$= \varphi\left(S(\alpha)f, \alpha, (S(\alpha)f)^*, 2\pi - \alpha\right)$$

$$\overset{T.I.}{=} \varphi\left(f, 0, (S(2\alpha)f)^*, 2\pi - 2\alpha\right)$$

$$= \varepsilon\ \varphi\left((S(2\alpha)f)^*, 2\pi - 2\alpha, f, 0\right)$$

$$\overset{\alpha \to \pi}{=} \varepsilon S(2\pi)\ \varphi(f^*, 0+, f, 0-)$$

$$= \varepsilon S(2\pi) \underbrace{\langle \Psi^*(f,0)\Omega, \Psi^*(f,0)\Omega \rangle}_{\geq 0}$$

$$\Longrightarrow \quad \varepsilon S(2\pi) = 1$$

$$S(2\pi) = (-1)^{2s_\Psi} \quad \Longrightarrow$$

$$\varepsilon \equiv \varepsilon_\Psi = (-1)^{2s_\Psi}$$

spin-statistics

$$\Updownarrow$$

KMS:

$$\varphi(f, 0, f^*, 0)$$

$$= \varepsilon_\Psi \varphi(f^*, 0, f, 0)$$

$$= \varphi(f^*, 0, S(2\pi)f, 2\pi)$$

2π-KMS for conj. by $e^{i\alpha M}$

Vacuum Ω is 2π - KMS state for Lorentz boosts $e^{i\alpha M}$.

Corresp. anti-unitary J, d <u>even</u>, is

$$J = PCT$$

QFT on non-trivial gravitational back-grounds, e.g. Schwarzschild, de Sitter, AdS, ...

Space-time M_d adm. complexification, M_d^c.

M_d^c has real Lorentzian section M_d, real Riemannian section M_d^E.

$M_d \leftrightarrow$ Killing symmetries (G, K, σ) symm. space

$M_d^E \leftrightarrow$ Killing symmetries (G^E, K, σ)

$$\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{m}, \quad \mathfrak{g}^E = \mathfrak{k} \oplus \mathfrak{m}$$

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$$

$$[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$$

$$\sigma|_{\mathfrak{k}} = id, \quad \sigma|_{\mathfrak{m}} = -id$$

$r : M_d^E \longrightarrow M_d^E$ involution (reflection), N: fixed-point set of r (imbedded submanifold of M_d^E)

K: Killing symmetries of N

$$r \circ g = g, \quad g \in K,$$

but

$$r \circ g = g^{-1},$$

for $g = \exp M$, $M \in \mathfrak{m}$,

i.e.,

$$r \circ g = \sigma(g)$$

Schwinger fns. over M_d^E, (G^E, K, σ) invariance; reflection positivity (r.!)

$\longrightarrow$ $\mathcal{H}$, Ω, unitary rep. of (G, K, σ) on $\mathcal{H}$, ...

T max. torus in G^E

T_m sub-torus of T gen. by elements in m.

Schwinger functions assumed to satisfy KMS w.r. to $T_m \longrightarrow$ PCT, "spin-statistics"

$\rightarrow$ p. 39 - 41